Limits on the Neutrino Mass and Mixing Angle from Pion and Lepton Decays

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Abstract

Motivated by a recent rather surprising conclusion based on the 1992 PDG data on the pion, kaon and lepton decays that if three generations of neutrinos are assumed to be massive and mixed, the heaviest neutrino, ν_3, could have a mass in the range, 155 MeV $\lesssim m_3 \lesssim$ 225 MeV, we have analyzed the latest 1995 data on the leptonic decays of pion, μ and τ with the assumption that three generations of neutrinos are massive and mixed. It is shown that when the radiative corrections are included and the constraint from partial decay widths is imposed, the 1995 data are consistent with three massless neutrinos with no mixing. Various limits on the neutrino mass and mixing angle implied by the 1995 data are presented together with a critique of the previous analysis.

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I. INTRODUCTION

In a series of seminal papers [1] in the early 1980's, Shrock proposed a wide range of experimental methods to obtain possible limits on the neutrino mass and associated mixing, all based on the precision analysis of the weak interaction data on the decays of the pion, kaon and charged leptons, μ and τ. In these papers, the comprehensive analysis of decay rates and branching ratios of the lepton and meson decays was carried out using the theoretical framework with three massive neutrinos and associated mixing. (It is interesting to note that as early as in 1961 Bahcall and Curtis [2] proposed a similar method based on pion and muon decays, even before the discovery of ν_μ!) At that time, however, available data were not accurate enough to provide any significant results on the limits on the neutrino mass and mixing angle, in the sense that the limit on the mixing angles were restricted mostly for large values of the neutrino masses. In the latter works [3,4] the limits were further improved.

The best known and often quoted limits on the neutrino mass still come from the analysis of spectral shapes in the Kurie plots or other decay kinematics [5]:

$$
\begin{aligned}
m_1 &\equiv m(\nu_1) \lesssim 5 \text{ eV [4]} \\
m_2 &\equiv m(\nu_2) \lesssim 270 \text{ KeV [4]} \\
m_3 &\equiv m(\nu_3) \lesssim 24 \text{ MeV [6]}.
\end{aligned}
\tag{1}
$$

The results from these analyses are always presented with the assumption of neutrinos with no mixing. A full analysis of the spectral shapes with three massive neutrinos with mixing is very much involved and so far no such analysis with satisfactory accuracy has been carried out.

In the meantime, much attention has been focused on entirely different approaches in which neutrino mass and associated mixing can be probed indirectly by searching for neutrino oscillation phenomena. The recent activities in this approach include the experimental search for reactor and accelerator neutrino oscillations and the study of the solar and atmospheric neutrinos. Although very intriguing indications of massive neutrinos with mixing have recently been hinted in the solar and atmospheric neutrino experiments and in the LSND experiment, a definitive answer from these experiments is yet to come.

Recently, Peres, Pleitez and Zukanovich Funchal (PPZ) [7] carried out a comprehensive analysis of the existing data on the meson and lepton weak decays assuming that three generations of neutrinos are massive and mixed. Their analysis was based on the 1992 Particle Data Group (PDG) data [8] combined with the latest (in 1993) data on τ decays [9]. The results are quite surprising in that the 1992 data on the decay branching ratios would be consistent with a finite mass for ν_3, i.e. 155 MeV $\lesssim m_3 \lesssim$ 225 MeV. This mass range is significantly larger than the most recent upper limit, $m_3 \lesssim 24$ MeV that was obtained from a kinematical analysis of the τ decay into five or six pions and ν_τ with no mixing. Moreover, PPZ found that the mixing angle β which represents mixing between ν_1 and ν_3 is also finite (11$\sim$12 degrees) whereas the mixing angle γ between ν_2 and ν_3 was bounded from above, thus allowing zero mixing angle.

Motivated by these rather surprising results, we have carried out a similar analysis of the decay rates and branching ratios of the leptonic decays of the pion, μ and τ with the

assumption of massive neutrinos with mixing. First we have repeated the PPZ analysis with the same set of the data (1992 PDG data) and with the assumptions used by PPZ, confirming their results. However, we have also found that their results are significantly modified when the constraint coming from the partial decay rates, which PPZ did not use, is imposed. The constraint imposed by the decay rate is not an independent one. Instead, it ensures that possible fortuitous cancellations in the ratios would not lead to erroneous conclusions. Furthermore, we have found that the radiative corrections, which PPZ also ignored, are quite important because the accuracy of the data more than warrants the inclusion of the radiative corrections in a precision analysis such as this. A similar analysis using the latest 1995 data shows that the 1995 data is consistent with the picture of three massless neutrinos with no mixing. We have been able to set various limits on the neutrino masses and mixing angles.

The plan of our paper is as follows. In Section II, we list the formulas relevant to our analysis. (Some details are put in Appendix.) All the data used in our analysis are collected in Section III, including the data of 1992 for comparison. In Section IV, we present details on our reexamination and critique of the PPZ analysis. In particular, we discuss here what would happen to the PPZ conclusions based on the 1992 PDG data, when the radiative corrections are included and the constraint from the partial decay rates is imposed. New results based on the 1995 data are presented in Section V, and a summary and conclusions are given in Section VI.

II. FORMULAS WITH THREE GENERATION MIXING

The mixing matrix V in the lepton sector which relates the (weak) interaction eigenfields ν_α ($\alpha = e, \mu, \tau$) to the mass eigenfields ν_i ($i = 1, 2, 3$) is given by

$$\nu_\alpha = \sum_{i=1}^{3} V_{\alpha i} \nu_i \tag{2}$$

We parameterize the mixing matrix V using the Maiani representation [10] of the mixing matrix U in the quark sector, with CP–violating phase set to zero, i.e.

$$V = \begin{pmatrix} c_\theta c_\beta & s_\theta c_\beta & s_\beta \\ -s_\theta c_\gamma - c_\theta s_\gamma s_\beta & c_\theta c_\gamma - s_\theta s_\gamma s_\beta & s_\gamma c_\beta \\ s_\theta s_\gamma - c_\theta c_\gamma s_\beta & -c_\theta s_\gamma - s_\theta c_\gamma s_\beta & c_\gamma c_\beta \end{pmatrix}. \tag{3}$$

where $s_\theta \equiv \sin\theta$, $c_\theta \equiv \cos\theta$, $s_\beta \equiv \sin\beta$ .... In Eq.(3), the angle θ refers to mixing between ν_1 and ν_2, β to ν_1 and ν_3 and γ to ν_2 and ν_3, respectively.

In our analysis, we will discuss the limits on neutrino masses and mixing angles which can be inferred from purely leptonic decays of pion and leptons (μ and τ). That is, the decay rates to be used are

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$$\Gamma(\pi \longrightarrow e\bar{\nu}_e), \quad \Gamma(\pi \longrightarrow \mu\bar{\nu}_\mu),$$
$$\Gamma(\mu \longrightarrow e\bar{\nu}_e\nu_\mu),$$
$$\Gamma(\tau \longrightarrow e\bar{\nu}_e\nu_\tau), \quad \Gamma(\tau \longrightarrow \mu\bar{\nu}_\mu\nu_\tau).$$

These are the best known experimental quantities which do not involve hadrons in the final states, hence introducing no further unnecessary complications in the calculation of decay widths. We will not consider the K decays, even though some experimental determination of its decay widths into leptons are almost as good as those of the pion. Its properties are quite similar to those of the pion and its data do not provide any additional (or critical) information.

Here, we briefly summarize the formulas to be used. (Details are given in Appendix.) For the pion, the decay rate into two leptons in a general case of three massive neutrinos with mixing is given by

$$\Gamma(\pi \longrightarrow l\bar{\nu}_l) = \frac{G^2 f_\pi^2 U_{ud}^2 m_\pi^3}{8\pi} \cdot \mathcal{R}_{\pi l} \cdot \sum_{i=1}^{3} |V_{li}|^2 P_i^{\pi l}, \tag{4}$$

where G is the Fermi constant (see comment below), m_π is the pion mass, f_π is the pion decay constant and U_{ud} is the ud component of the mixing matrix in the quark sector. The matrix element–phase space function, $P_i^{\pi l}$, denotes the quantity of our interest which contains part of the matrix element and the entire phase space, and it depends on the neutrino masses, m_i ($i = 1, 2, 3$), as well as on the pion and lepton masses. $\mathcal{R}_{\pi l}$ is a factor that represents the radiative corrections to the process. We stress here that $\mathcal{R}_{\pi l}$ depends on the pion mass as well as on the lepton mass. The complete expressions for $P_i^{\pi l}$ and $\mathcal{R}_{\pi l}$ are given in Appendix. We wish to emphasize here that the value of $f_\pi = (130.7 \pm 0.1 \pm 0.36)$ MeV quoted in PDG data set is obtained from the decay $\pi \longrightarrow \mu\nu_\mu + \mu\nu_\mu\gamma$ under the hypothesis that the neutrinos are massless with no mixing. For massive and mixed neutrinos, the above value represents the quantity $[f_\pi^2 \sum_{i=1}^{3} |V_{\mu i}|^2 P_i^{\pi\mu}/P_0^{\pi\mu}]^{1/2}$ rather than f_π, where $P_0^{\pi\mu}$ is the matrix element–phase space function for massless neutrinos. For this reason, in the following, we will consider, as was done by PPZ, only the ratio of the two leptonic decay widths of the pion, in order to cancel out the dependence on the unknown quantity f_π.

The decay width for a lepton decaying into three leptons is given by

$$\Gamma(l' \longrightarrow l\bar{\nu}_l\nu_{l'}) = \frac{G^2 m_{l'}^5}{192\pi^3} \cdot \mathcal{R}_{l'} \cdot \sum_{i,j=1}^{3} |V_{l'i}|^2 |V_{lj}|^2 P_{ij}^{l'l}, \tag{5}$$

where $m_{l'}$ is the mass of the decaying particle. Again, $P_{ij}^{l'l}$ is the matrix element–phase space function which depends on the masses of all the particles involved in the decay process. The leading radiative corrections are denoted by $\mathcal{R}_{l'}$; they depend only on the mass of the decaying lepton. Also, the expressions for $P_{ij}^{l'l}$ and $\mathcal{R}_{l'}$ are collected in Appendix.

Again, it is crucial to emphasize, as noted by PPZ, that the experimental value of G_μ quoted in the PDG data set becomes the Fermi constant G only for the massless neutrinos, as implied by the Standard Model. Since new physics beyond the Standard Model is what we wish to investigate, the coupling constant G_μ that one measures in the muon decay should be interpreted as

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$$G_\mu^2 = \left[\frac{G^2}{P_{00}^{\mu e}}\right] \times \sum_{i,j=1}^{3} |V_{\mu i}|^2 |V_{ej}|^2 P_{ij}^{\mu e}$$

$$\begin{aligned}
= \left[\frac{G^2}{P_{00}^{\mu e}}\right] \times \Big\{ &c_\theta^2 c_\beta^2 (s_\theta c_\gamma + c_\theta s_\gamma s_\beta)^2 P_{11}^{\mu e} \\
&+ c_\beta^2 [c_\theta^2 (c_\theta c_\gamma + s_\theta s_\gamma s_\beta)^2 + s_\theta^2 (s_\theta c_\gamma + c_\theta s_\gamma s_\beta)^2] P_{12}^{\mu e} \\
&+ [c_\beta^2 (c_\theta^2 c_\beta^2 s_\gamma^2) + s_\beta^2 (s_\theta c_\gamma + c_\theta s_\gamma s_\beta)^2] P_{13}^{\mu e} \\
&+ s_\theta^2 c_\beta^2 (c_\theta c_\gamma + s_\theta s_\gamma s_\beta)^2 P_{22}^{\mu e} \\
&+ [c_\beta^2 (s_\theta^2 c_\beta^2 s_\gamma^2) + s_\beta^2 (c_\theta c_\gamma + s_\theta s_\gamma s_\beta)^2] P_{23}^{\mu e} \\
&+ s_\beta^2 c_\beta^2 s_\gamma^2 P_{33}^{\mu e} \Big\}
\end{aligned}$$

$$\longrightarrow G^2 \quad \text{as } m_1, m_2, m_3 \longrightarrow 0$$

(6)

where $P_{00}^{\mu e}$ is the matrix element–phase space function for massless neutrinos. Therefore, in a general case of massive neutrinos with mixing, the weak coupling constant G, which enters in the calculation of all the weak processes like e.g. Eq(4) and Eq.(5), is *not* directly measured from the muon decay. The quoted number $G_\mu = (1.16639 \pm 0.00002) \cdot 10^{-5}$ GeV^{-2} is valid only for massless neutrinos.

An alternative way to obtain G comes from the Standard Model of electroweak interactions, where G, the fine–structure constant α, the weak mixing angle θ_W and the W boson mass are related as [4,11]

$$G = \frac{\pi\alpha}{\sqrt{2} m_W^2 \sin^2 \theta_W (1 - \Delta r)}. \tag{7}$$

The above equation relates the low-energy quantities G and α to the quantities defined at the electroweak scale, $\sin^2 \theta_W$ and m_W. The radiative effects are taken into account in Δr, which in the first order in α can be cast in the following way

$$\Delta r = \Delta r_\alpha + \Delta r_t + \Delta r_f + \Delta r_q + \Delta r_b \tag{8}$$

The first term in Eq.(8) describes the running of α from the zero–momentum scale to the electroweak scale, the second term is due to the radiative contribution of the top, Δr_f and Δr_q are contributions coming from the leptons and quarks (except the top) and the last one is due to the Higgs–boson loops. The actual form of the radiative correction Δr depends on the renormalization scheme adopted to perform the calculations.

Hence, in the Standard Model only three out of the four quantities G, α, $\sin^2 \theta_W$ and m_W are independent. Since the Fermi constant (for massless neutrinos), the fine–structure constant and the Weinberg angle are the best known quantities, the standard practice is to consider them as being independent, and then use Eq.(7) to predict the W mass.

The aim of our analysis is to investigate the possibility that neutrinos are massive and mixed, therefore G, as was discussed, is different from G_μ and hence should be treated as unknown. Namely, we reverse the standard procedure and use Eq.(7) to calculate the value and the allowed 1–σ range for the Fermi constant, using α, $\sin^2 \theta_W$ and m_W as input parameters. Obviously we expect the accuracy of the value G to be rather poor, i.e. of the same order of the experimental error on m_W. It is encouraging, however, that the latest measurement on the W mass are at the level of 2% [12].

Special caution must be exercised in choosing the value of $\sin^2 \theta_W$ to be used in Eq.(7). This parameter has been obtained to a very good accuracy by a number of different experiments (see, e.g. Ref. [4] and references quoted therein, for a summary of the different techniques used to extract $\sin^2 \theta_W$). All these methods involve the measurement of cross sections, and the fits of the data (from which $\sin^2 \theta_W$ is obtained) are made using G for massless neutrinos ($G = G_\mu$) as an input parameter (directly in the cross section evaluations, or indirectly through the interference terms in the calculations of the asymmetries). This means that the best mean value of $\sin^2 \theta_W$ quoted in the PDG data cannot be used in Eq.(7), because it is not independent of G itself. Therefore, we perform our calculation of G in the on–shell scheme of renormalization [11], where the following relation is defined to be true at all orders in the perturbation theory

$$\sin^2 \theta_W = 1 - \frac{m_W^2}{m_Z^2}. \tag{9}$$

In this way $\sin^2 \theta_W$ is obtained independently of G, even though its uncertainties will become larger than that of the best mean value in PDG data due to the uncertainties in the weak boson masses. (A further comment: strictly speaking, the determination of m_W and m_Z could, in principle, be also affected by the value of G. In fact, the usual way to determine the weak boson masses involves a global fit of a number of physical quantities and G is again taken as an input parameter of the fit [13]. Therefore, the actual value of G could influence very slightly the determination of m_W and m_Z. However, the correlation between the value of G and the position of the poles in the $W-$ and Z–production cross–sections (basically the weak boson masses) cannot be substantial. For this reason we assume the quoted measured values of m_W and m_Z as being independent of G.)

In summary, we have chosen to determine the value of the Fermi constant and its $1-\sigma$ allowed interval by means of the Standard Model relation Eq.(7), in the framework of the on–shell renormalization scheme where $\sin^2 \theta_W$ is defined by Eq.(9). The actual expressions for the radiative corrections Δr in the on–shell scheme can be found in Ref. [11]. The top contribution Δr_t, not contained in the original paper by Sirlin, is [4]

$$\Delta r_t = -\frac{3\alpha}{16\pi \sin^2 \theta_W} \frac{m_t^2}{m_W^2} \frac{1}{\tan^2 \theta_W}. \tag{10}$$

Hence our input parameters are: m_Z, m_W and α. (Obviously, the low–energy value of the fine structure constant, obtained from the quantum Hall effect, is independent from G). Some additional input parameters, which enter in the radiative corrections Δr, are the mass of the top quark m_t and the Higgs boson mass m_H. For the top mass, we use the recent CDF measurement $m_t = (176 \pm 18)$ GeV [14], both for the 1992 and 1995 data sets. The Higgs mass is varied in the interval (60 GeV, 1 TeV).

In the case of the 1992 data set, we use the 1992 PDG values for m_Z and m_W. The results are (here and hereafter, the errors are propagated quadratically)

$$\begin{aligned} \sin^2 \theta_W &= 0.2258 \pm 0.0050 \\ \Delta r &= 0.0402 \pm 0.0088 \\ G &= (1.162 \pm 0.029) \cdot 10^{-5} \text{ GeV}^{-2} \end{aligned} \tag{11}$$

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For the 1995 data, in order to reduce as much as possible the uncertainties in the determination of G, we use the latest data available: $m_Z = (91.1884 \pm 0.0022)$ GeV [15] and $m_W = (80.410 \pm 0.180)$ GeV [12]. This gives

$$\begin{aligned}
\sin^2 \theta_W &= 0.2224 \pm 0.0035 \\
\Delta r &= 0.0396 \pm 0.0088 \\
G &= (1.174 \pm 0.022) \cdot 10^{-5} \text{ GeV}^{-2}
\end{aligned} \tag{12}$$

The values of G which we will use in the evaluation of the leptonic decay widths are those given in the two previous equations. Unfortunately, the errors on G (of the order of percent) are much worse than the errors on G_μ which are of the order of 10^{-5}. We stress that since we are interested in the investigation of massive and mixed neutrinos, the method discussed above turns out to be the only viable way to determine the Fermi constant. Unfortunately, however, this procedure determines G with errors of a few percent level and thus makes the accuracy in the calculation of the decay widths rather poor. Nevertheless, as will be seen in the following, it is still possible to use them as an important constraint on the neutrino parameters (mainly on the neutrino masses). In fact, a deviation (if any) of G_μ from the Standard Model value of G given in Eq.(7) signals the massive and mixed neutrinos.

Measured experimentally are the branching ratios of the above decay processes. The branching ratios are simply related to the previously defined quantities as

$$\text{BR}(\pi \longrightarrow l\bar{\nu}_l) = \frac{\Gamma(\pi \longrightarrow l\bar{\nu}_l)}{\Gamma_\pi} = \tau_\pi \cdot \Gamma(\pi \longrightarrow l\bar{\nu}_l) \tag{13}$$

$$\text{BR}(l' \longrightarrow l\bar{\nu}_l\nu_{l'}) = \frac{\Gamma(l' \longrightarrow l\bar{\nu}_l\nu_{l'})}{\Gamma_{l'}} = \tau_{l'} \cdot \Gamma(l' \longrightarrow l\bar{\nu}_l\nu_{l'}) \tag{14}$$

where Γ_π and $\Gamma_{l'}$ are the total widths of the pion and the decaying lepton, respectively, and τ_π and $\tau_{l'}$ are the corresponding lifetimes. In order to directly extract information about the neutrino mass and mixing, instead of the lepton decay widths themselves, we use the following quantities which are simply proportional to the decay widths, with common constants such as G and $m_{l'}$ removed,

$$\overline{\Gamma}^{l'l} = \alpha_{l'}\text{BR}(l' \longrightarrow l\bar{\nu}_l\nu_{l'}) = \mathcal{R}_{l'} \cdot \sum_{i,j=1}^{3} |V_{l'i}|^2 |V_{lj}|^2 P_{ij}^{l'l}, \tag{15}$$

where

$$\alpha_{l'} = \frac{192\pi^3}{G^2 m_{l'}^5 \tau_{l'}}. \tag{16}$$

From the experimental values of the BR's and the physical quantities defined in Eqs. (16), we will first calculate the 1–σ allowed experimental ranges for the $\overline{\Gamma}$'s. These ranges will then be compared with the calculated values of the $\overline{\Gamma}$'s by varying the neutrino masses and mixing angles. This would limit the neutrino masses and mixing angles.

We will follow the same procedure for some ratios of the $\overline{\Gamma}$'s and the BR's. Our choice of the ratios is

$$R_{\pi\mu}^{\pi e} \equiv \frac{\mathrm{BR}(\pi \longrightarrow e\bar{\nu}_e)}{\mathrm{BR}(\pi \longrightarrow \mu\bar{\nu}_\mu)} = \frac{\mathcal{R}_{\pi e} \cdot \sum_i |V_{ei}|^2 P_i^{\pi e}}{\mathcal{R}_{\pi\mu} \cdot \sum_i |V_{\mu i}|^2 P_i^{\pi\mu}}, \tag{17}$$

$$R_{l_2' l_2}^{l_1' l_1} \equiv \frac{\overline{\Gamma}^{l_1' l_1}}{\overline{\Gamma}^{l_2' l_2}} = \left(\frac{m_{l_2'}^5 \tau_{l_2'}}{m_{l_1'}^5 \tau_{l_1'}} \right) \frac{\mathrm{BR}(l_1' \longrightarrow l_1 \bar{\nu}_{l_1} \nu_{l_1'})}{\mathrm{BR}(l_2' \longrightarrow l_2 \bar{\nu}_{l_2} \nu_{l_2'})} = \frac{\mathcal{R}_{l_1'} \cdot \sum_{i,j} |V_{l_1' i}|^2 |V_{l_1 j}|^2 P_{ij}^{l_1' l_1}}{\mathcal{R}_{l_2'} \cdot \sum_{i,j} |V_{l_2' i}|^2 |V_{l_2 j}|^2 P_{ij}^{l_2' l_2}}. \tag{18}$$

In the case of lepton decays, we will perform our analysis using the two ratios, $R_{\tau e}^{\mu e}$ and $R_{\tau\mu}^{\tau e}$. The use of ratios alone is indeed simpler because uncertainties in some constant quantities are cancelled out, but some changes in the numerator and the denominator coming from phase space and mixing angles may partially be compensated. Therefore one must check that the calculated single partial decay widths do not lie outside the experimental allowed ranges. This is why we will add as an additional constraint also the three leptonic decay widths. Summarizing, we will use the following quantities as constraint: $\overline{\Gamma}^{\mu e}$, $\overline{\Gamma}^{\tau e}$, $\overline{\Gamma}^{\tau\mu}$, $R_{\pi\mu}^{\pi e}$, $R_{\tau e}^{\mu e}$ and $R_{\tau\mu}^{\tau e}$

When we evaluate the $\overline{\Gamma}$'s and the R's from the experimental values, we propagate the errors quadratically. In the calculations, we use the central values of masses of the particles involved. All the calculations are carried out at 1–σ level. Also, it is to be noted that the radiative correction factors $\mathcal{R}$'s are included in the quantities $\overline{\Gamma}$'s; they are cancelled out in $R_{\tau\mu}^{\tau e}$ but not in $R_{\tau e}^{\mu e}$ and $R_{\pi\mu}^{\pi e}$.

III. NUMERICAL INPUTS

We have listed, in Table 1, all the latest (1995 data) experimental inputs that will be used in our analysis [4,16]. In order to compare our new analysis with the previous one by PPZ based on the 1992 PDG data, we have also listed the 1992 PDG data [8] in Table 1. As can be seen in Table 1, the entries with asterisks signify those with noticeable changes from the 1992 data to the 1995 data. In particular, they include all the data on τ decays and $\pi \rightarrow e\bar{\nu}_e$. For the sake of comparison, in Table 1 the value of both G_μ and G are included. (We note that, strictly speaking, m_τ quoted in Table 1 is for massless ν_τ. However, in this case, the use of m_τ in Table 1 does not introduce any significant modification.)

We should notice that the errors on the $\overline{\Gamma}$'s are quite large (of the order of a few percent) due to the uncertainties in the Fermi constant G, as discussed in the previous Section. Nevertheless, the inclusion of the $\overline{\Gamma}$ constraint turns out to be effective in limiting the allowed intervals for the neutrino parameters, in particular the masses, as will be shown in the following Sections.

Listed below are the *calculated* values of $\overline{\Gamma}$'s and R's using the latest mass values for the decaying particles and charged leptons under the assumption that neutrinos are massless with no mixing. In the case of $\overline{\Gamma}$'s, the first numbers on the right–hand side are the values without the radiative corrections and the second numbers represent the radiative corrections. All the values are in agreement with the 1995 data in Table 1 within 1–σ, implying that the 1995 data are consistent with the lepton sector with massless neutrinos with no mixing.

Calculated values for massless neutrinos with no mixing

$$\overline{\Gamma}^{\mu e} = 0.999813 \cdot 0.995797 = 0.995611 \qquad R^{\pi e}_{\pi\mu} = 1.233 \cdot 10^{-4}$$
$$\overline{\Gamma}^{\tau e} = 1.00 \cdot 0.996 = 0.996 \qquad R^{\mu e}_{\tau e} = 0.9995$$
$$\overline{\Gamma}^{\tau\mu} = 0.972 \cdot 0.996 = 0.968 \qquad R^{\tau e}_{\tau\mu} = 1.028$$

(The uncertainties in the above numbers due to the experimental errors of the quantities which enter in their calculations are always less than 0.01%).

The conclusion based on the above numbers that the experimental data (1995) are consistent with the assumption of massless neutrinos with no mixing is in sharp contrast to the result coming from the 1992 PDG data, as obtained by PPZ. In order to further examine 1992 data set, we have repeated the calculations using the 1992 PDG data both with and without the inclusion of the radiative corrections. We list, in the following, the results of the *calculated* values based on the 1992 PDG data in Table 1 with the assumption that neutrinos are massless with no mixing.

Calculated values for massless neutrinos with no mixing
(1992 PDG data set)

$$\overline{\Gamma}^{\mu e} = 0.999813 \cdot 0.995797 = 0.995611$$
$$\overline{\Gamma}^{\tau e} = 1.00 \cdot 0.996 = 0.996$$
$$\overline{\Gamma}^{\tau\mu} = 0.973 \cdot 0.996 = 0.969$$

$$R^{\pi e}_{\pi\mu} = 1.233 \cdot 10^{-4} \qquad (R^{\pi e}_{\pi\mu} = 1.283 \cdot 10^{-4} \qquad \text{without RC})$$
$$R^{\mu e}_{\tau e} = 0.9995 \qquad (R^{\mu e}_{\tau e} = 0.9998 \qquad \text{without RC})$$
$$R^{\tau e}_{\tau\mu} = 1.028 \qquad (R^{\tau e}_{\tau\mu} = 1.028 \qquad \text{without RC})$$

(Again, the uncertainties in the above numbers are also less than 0.01%).

Note that the calculated values of the following quantities do *not* lie in their corresponding experimental 1–σ ranges: $R^{\pi e}_{\pi\mu}$ and $R^{\mu e}_{\tau e}$. This implies that the 1992 PDG data are indeed incompatible, at least within 1–σ, with the assumption that neutrinos are massless with no mixing. If the radiative corrections are not included, $\overline{\Gamma}^{\tau e}$ is not compatible, also.

IV. REEXAMINATION OF PREVIOUS ANALYSIS

In this Section we will first reexamine in detail the result of the PPZ analysis and then present the results of our new analysis. As mentioned already, the PPZ analysis was based on the 1992 PDG data, improved by the latest (at that time) determination of the τ mass. In order to cancel out the dependence of the decay widths on some parameters (the pion mass, the quark mixing angle U_{ud}, the pion decay constant, the Fermi constant and the muon or tau mass), PPZ considered only the ratios of the partial decay widths; $R^{\pi e}_{\pi\mu}$, $R^{\mu e}_{\tau e}$ and $R^{\tau e}_{\tau\mu}$. They did not take into account radiative corrections for these processes under the assumption that the radiative corrections of order of several percents are of no importance

(note that the radiative corrections do not cancel each other in the ratios $R_{\pi\mu}^{\pi e}$ and $R_{\tau e}^{\mu e}$).
The PPZ analysis was performed in the case of one massive neutrino (ν_3) and two almost
degenerate very light ($m_1 \sim m_2 \ll m_3$) ones. Their main result is that ν_3 could have a mass
in the interval

$$155 \text{ MeV} \lesssim m_3 \lesssim 800 \text{ MeV}. \tag{19}$$

That is, all the 1992 PDG data could be fitted with m_3 in the above range. They then
improved the upper limit in Eq.(19) by taking into account the constraint coming from the
Z invisible width. The resulting allowed interval was

$$155 \text{ MeV} \lesssim m_3 \lesssim 225 \text{ MeV}. \tag{20}$$

Given the values of m_3 inside this range, the PPZ analysis also showed that one of the
mixing angles (namely β) was constrained to a finite range which did not include $\beta = 0$.
They obtained, for $m_3 = 165$ GeV,

$$11.54° \lesssim \beta \lesssim 12.82°. \tag{21}$$

Although the above mass and mixing angle intervals are allowed by the ratios R, one
must make sure that the same allowed ranges do not violate the experimental partial decay
widths. This turns out to be the case for the mass range $m_3 \gtrsim 215$ MeV, as will be shown
in Section IV–A.

In the following we will examine what would happen to the above PPZ conclusions if
1) radiative corrections are taken into account, 2) the constraint from $\overline{\Gamma}$'s (decay widths)
is imposed and 3) 1995 data are used together with the $\overline{\Gamma}$ constraint and the radiative
corrections. Specifically, we will show that due to the accuracy of the present data, it
is important to include radiative corrections and that the use of ratios R's alone without
checking the partial decay widths (i.e. $\overline{\Gamma}$'s) could lead to overestimates of the allowed interval
of the neutrino parameters.

A. Allowed Range for Mass

In Fig.1 we have plotted the ranges for the values of m_3 which are forbidden (denoted
by solid lines) by the ratios R's, by the $\overline{\Gamma}$'s and by the combination of the two. The heavy
solid line represents the allowed region. In this plot, it is assumed as in the case of the PPZ
that the other mass parameters are very small ($m_1 \simeq m_2 \ll m_3$) and the mixing angles
are varied over the maximum interval $(0, \pi/2)$. The figure refers to three cases: 1992 PDG
data, without radiative corrections (RC), denoted by 92, 1992 PDG data with RC, denoted
by 92RC and 1995 data with RC, denoted by 95RC, respectively. For each case we present
the three results, one with the R's, one with the $\overline{\Gamma}$'s and one with R's and $\overline{\Gamma}$'s combined.
For 92, there is an allowed region for m_3 which does not include $m_3 = 0$ when R's alone are
used. This is the PPZ result. Most values of m_3 inside this allowed region, however, violate
the limits on the $\overline{\Gamma}$'s, as can be seen in Fig.1. It has to be emphasized that the range for m_3
allowed by the $\overline{\Gamma}$'s alone does not include $m_3 = 0$, also.

If we combine the two results, we find the following allowed range for m_3

$$178 \text{ MeV} \lesssim m_3 \lesssim 215 \text{ MeV}. \tag{22}$$

The inclusion of radiative corrections changes the picture dramatically. As can be seen in Fig.1 (92RC), the region allowed by R's is considerably enlarged. Also the $\overline{\Gamma}$ constraint is modified in an important way, i.e. all the region of lower masses is now allowed, including $m_3 = 0$. When we combine the two constraints $(R + \overline{\Gamma})$ the PDG 92 data set is consistent with the following finite (not including zero) mass range for m_3

$$140 \text{ MeV} \lesssim m_3 \lesssim 210 \text{ MeV}. \tag{23}$$

signaling new physics beyond the Standard Model. It is, therefore, extremely interesting to repeat the analysis with the 1995 data. It is to be noted that an agreement of the above result with the PPZ's (Eq.(20)) is purely accidental.

As can be seen in 95RC in Fig 1, R's alone allow two different (disconnected) regions, one of which includes $m_3 = 0$. The inclusion of $\overline{\Gamma}$'s restricts these intervals to the region of lower masses. The allowed regions resulting from the combined (both R's and $\overline{\Gamma}$'s) analyses based on the 1995 data with radiative corrections is (for $m_1 \simeq m_2 \ll m_3$)

$$m_3 \lesssim 70 \text{ MeV} \quad \text{and} \quad 140 \text{ MeV} \lesssim m_3 \lesssim 149 \text{ MeV}. \tag{24}$$

The above result is rather insensitive to the choice of m_1 and m_2. For example, for $m_1 \lesssim 20 \text{ KeV}$ and $m_2 \lesssim 1 \text{ MeV}$, the above result remains unchanged. Only if m_2 is of the order of a few MeV all the region between $m_3 = 0$ and $m_3 \simeq 149 \text{ MeV}$ is allowed. Therefore, we conclude, based on the 1995 data, that the 1992 PDG data were internally inconsistent because of the poor data on τ. Moreover, the accuracy of the current data or even the 1992 PDG data warrants the inclusion of the radiative corrections for any precision analysis. Furthermore, one can see that the use of the R's alone without the constraint from $\overline{\Gamma}$'s can give rise to overestimated allowed regions.

B. Allowed Range for Mixing Angles

So far we have reexamined and discussed the PPZ using the 1992 PDG data. We have also carried out a similar analysis using the 1995 data with a conclusion that there is an allowed window for m_3 including $m_3 = 0$, as well as an isolated range of m_3 (for $m_1 \simeq m_2 \ll m_3$). In this Sub–Section we carry out a similar analysis for mixing angles. The PPZ concludes, based on the ratios alone from 1992 PDG data, that the angle β has a finite allowed range, $11.54° \lesssim \beta \lesssim 12.82°$, whereas the angle γ is restricted to $\gamma \lesssim 4.05°$, including zero. This allowed region is shown in the β–γ plane as an area filled with circles in Fig.2a. In addition, the constraint imposed by the $\overline{\Gamma}$'s alone is indicated by the dotted region. (It is to be pointed out that in Fig.2 the mass parameters are $m_1 \simeq m_2 \simeq 0$ and $m_3 = 200 \text{ MeV}$). The region allowed by the two constraints combined is denoted by the dark area.

Now, the inclusion of radiative corrections to the PPZ analysis leads to changes in the allowed regions of Fig.2a. The region allowed by R's alone is enlarged and moved towards the origin, but still it does not include the origin that corresponds to the case of neutrinos

without mixing. This is shown by circles in Fig.2b. Also shown in Fig.2b is the allowed region based on $\overline{\Gamma}$'s alone, with the radiative corrections included (dots). The darker area is the region allowed by the R's and $\overline{\Gamma}$'s combined. Therefore, the common allowed region is

$$4° \lesssim \beta \lesssim 10° \tag{25}$$
$$\gamma \lesssim 7.2° \tag{26}$$

When we use the 1995 data (with radiative corrections), the situation again changes dramatically. Fig.2c shows the allowed region in the β–γ plane which is obtained by using the ratios R alone (circles). The allowed region is shown to move farther away from the origin. The addition of the constraint coming from the $\overline{\Gamma}$'s completely washes out the region, i.e. no allowed region. This is self–evident because the mass $m_3 = 200$ MeV is not allowed by the 1995 data, as can be seen in Eq.(24). Thus, even in the case of the 1995 data, neglecting the $\overline{\Gamma}$ constraint could lead to erroneous conclusions.

V. ANALYSIS AND RESULTS

In this Section we present the results of a more detailed analysis of the limits that can be set on the neutrino masses and mixing angles by using the 1995 data and with the inclusion of radiative corrections. It is to be pointed out here that a complete, combined analysis of the masses (m_1, m_2 and m_3) and mixing angles (θ, β and γ) is very much involved and is beyond the scope of this paper. Even the presentation of results of such an analysis would be problematic. Therefore, we have simplified the analysis by fixing some parameters and varying others. In order to see various correlations among the masses and the mixing angles, we present several allowed regions in the two dimensional plots for several combinations of mass and mixing angle.

First, we present the absolute upper limits on the three neutrino masses, m_1, m_2, and m_3, independent of the values of mixing angles, θ, β and γ. We have obtained these limits by varying the mixing angles over the entire interval between 0 and $\pi/2$ and by taking into account all the constraint which we have discussed in the previous Section (R's and $\overline{\Gamma}$'s) and the radiative corrections. (It should be stated that we are <u>not</u> carrying out a statistical analysis of all the relevant data.) Instead of three dimensional plots, we present, in Fig.3, the allowed region (dotted area) in the m_1–m_2 plane and the allowed region in m_2–m_3 plot in Fig.4. From these plots, we can set the following *absolute* upper limits on the neutrino masses (based on $\overline{\Gamma}$ and R constraint, at 1–σ level)

$$\begin{aligned} m_1 &\lesssim 100 \text{ KeV}, \\ m_2 &\lesssim 7.5 \text{ MeV}, \\ m_3 &\lesssim 149 \text{ MeV}. \end{aligned} \tag{27}$$

where the limits on m_1 and m_2 are mainly due to the R's, whereas the limit on m_3 comes from the $\overline{\Gamma}$ constraint. The limit on m_3 has already been mentioned in the previous Section. Although the limit on m_1 is rather poor, the limit on m_2 is larger by a factor of less than thirty than the latest limit from the kinematical analysis of the $\pi \to \mu + \nu_\mu$ decay. Similarly, the limit on m_3 is larger only by a factor of six. It is quite interesting that the accuracy

of the present data on the decay rates and branching ratios is already sufficiently good enough to set limits on m_2 and m_3 which agree, in one order of magnitude, with the results from more involved kinematical determination. The important difference between the upper limits given in Eq.(27) and those in Eq.(1) is that the former is valid independently of mixing angles whereas the latter is valid only for the case of no mixing. Furthermore, the improvement in the data from 1992 is obvious from the conclusion that the upper limit on m_3 is set to 149 MeV and the limits are consistent with massless neutrinos with no mixing, implying the internal consistency of the data.

Next, we discuss some correlations among the masses and the mixing angles. The first example to be presented is the allowed region in the m_2–$\sin^2\theta$ plot shown in Fig.5. In this plot, we have set m_3 to be 24 MeV and ν_3 is assumed to be decoupled, due to its heavy mass, from ν_1 and ν_2 so that $\beta = \gamma = 0$. Also, for definiteness, we have taken $m_1 = 5$ eV, but the conclusion remains unchanged as long as m_1 is less than ~ 20KeV. The solid and dashed lines delimit allowed regions based on the use of $\overline{\Gamma}$'s and the constraint from R's, respectively. The allowed area is denoted by dots. Fig.5 shows that the low–angle regime is constrained mainly by $R_{\tau e}^{\mu e}$ whereas $R_{\pi\mu}^{\pi e}$ is more effective in limiting the large–angle area (say $\sin^2\theta \gtrsim 10^{-4}$). The allowed region shown in Fig.5 is insensitive to values of m_1 and m_3, as long as they are $m_1 \lesssim 20$ KeV and $m_3 \lesssim 50$ MeV. The so–called small– and large– angle solutions of the solar neutrino deficit based on the MSW effect, which requires $m_2^2 - m_1^2 \simeq 6 \times 10^{-6}$eV^2 and $\sin^2(2\theta) \simeq 7 \times 10^{-3}$, and $m_2^2 - m_1^2 \simeq 8 \times 10^{-6}$eV^2 and $\sin^2(2\theta) \simeq 0.6$, respectively, are well within the allowed region in this plot.

In the next example, we assume that ν_1 is too light to couple with ν_2 and ν_3. That is, only ν_2 and ν_3 are mixed with angle γ. We have also set $m_1 \ll m_2 = 270$ KeV. Fig.6 shows the allowed region in the m_3–$\sin^2\gamma$ plane. Here, the small– and large– angle regions are constrained by $R_{\tau e}^{\mu e}$, whereas the intermediate region ($\sin^2\gamma \sim 10^{-4}$–$10^{-2}$) is constrained by $R_{\pi\mu}^{\pi e}$. The neutrino oscillation solution of the atmospheric neutrino anomaly as observed by Kamiokande and others, which favors $m_3^2 - m_2^2 \simeq 10^{-2}$eV^2 and $\sin^2(2\theta) \simeq 1$, is well within the allowed region in Fig.6. The size and the shape of the allowed region are insensitive to the assumed values of m_1 and m_2 as long as $m_1 \lesssim 20$ KeV and $m_2 \lesssim 1$ MeV.

Although the case of a ν_1–ν_3 mixing is unnatural in the framework of natural hierarchy of neutrino masses, we present it in Fig.7. In this Figure the allowed region in the m_3 – $\sin^2\beta$ plane is shown. Here, as in the previous case we have set $m_1 \ll m_2 = 270$ KeV. In this case, the most restricted limit is always imposed by $R_{\pi\mu}^{\pi e}$. Again, the allowed region does not change significantly as long as $m_1 \lesssim 20$ KeV and $m_2 \lesssim 1$ MeV.

In the previous figures Figs.5–7 the limits were obtained in the special cases in which two of the three mixing angles were kept fixed at zero, i.e. only one pair of neutrinos is mixed. To show how sensitive these limits are to the fixed angles and correlations among the limits, we present in Figs.8–10 the cross sections (with one angle fixed) of the three–angle parameter space for fixed values of the masses, i.e. $\sin^2\gamma$– $\sin^2\beta$ in Fig.8, $\sin^2\gamma$–$\sin^2\theta$ in Fig.9 and $\sin^2\beta$– $\sin^2\theta$ in Fig.10, respectively. In each figure, the values of the masses are fixed as $m_1 \ll m_2 = 270$ KeV and $m_3 = 24$ MeV. It is interesting to note that for the above set of the masses, the most severely constrained angle is β, whereas the least constrained is γ. The limits are:

$$\sin^2\theta \lesssim 3.6 \cdot 10^{-3}$$
$$\sin^2\beta \lesssim 4.6 \cdot 10^{-7} \qquad (28)$$
$$\sin^2\gamma \lesssim 7.0 \cdot 10^{-3}.$$

As we decrease the values of the masses used, the allowed region in each figure increases, eventually covering the entire space. Hence, no meaningful limit can be obtained, as expected. In order to demonstrate this sensitivity, we have shown in Figs.8–10 the extended allowed regions (bounded by the dashed lines) for the following values of the masses: $m_1 \ll m_2 = 10$ KeV and $m_3 = 1$ MeV. In this case, the limits on the mixing angles substantially increase to

$$\sin^2\theta \lesssim 1$$
$$\sin^2\beta \lesssim 2.6 \cdot 10^{-4} \qquad (29)$$
$$\sin^2\gamma \lesssim 1.$$

VI. SUMMARY AND CONCLUSIONS

We have analyzed and compared the 1992 and 1995 data on the π, μ and τ decays in the framework of three generations of massive neutrinos with associated mixing. First we have confirmed the surprising result of PPZ based on the 1992 PDG data that when only the ratios, R's, are used without radiative corrections, the 1992 data are inconsistent with the picture of massless neutrinos with no mixing, signaling new physics beyond the Standard Model! More specifically, PPZ has shown that when m_1 and m_2 are assumed to be much less than m_3, m_3 is found to be within the interval 155 MeV $\lesssim m_3 \lesssim$ 225 MeV and the $\nu_1 - \nu_3$ mixing angle, β, in the interval, $11.54° \lesssim \beta \lesssim 12.84°$. This isolated allowed region survives even if we introduce the $\overline{\Gamma}$ constraint.

Again, the 1992 PDG data set with radiative corrections reproduces the results that agree qualitatively with those of PPZ. The allowed range of m_3 is 140 MeV $\lesssim m_3 \lesssim$ 210 MeV and the allowed mixing angles are $4° \lesssim \beta \lesssim 10°$ and $\gamma \lesssim 7.2°$. This clearly shows that the 1992 PDG data set suggests massive and mixed neutrino.

In order to see if this rather surprising result still remains valid or not with the improved data of 1995, we have carried out a comprehensive analysis of the 1995 data by using both the ratios, R's, and decay widths, $\overline{\Gamma}$'s, and by including the radiative corrections. The 1995 data are shown to be internally consistent, and furthermore consistent with the picture of massless neutrinos with no mixing.

Limits on the masses derived from the analysis are as follows:

$$m_1 \lesssim 100 \text{ KeV},$$
$$m_2 \lesssim 7.5 \text{ MeV}, \qquad (30)$$
$$m_3 \lesssim 149 \text{ MeV}.$$

These bounds on the masses are such that the imposed constraint (R's and $\overline{\Gamma}$'s) are fulfilled in their 1–σ intervals.

Although the above limits are less stringent than those in Eq.(1) from kinematical determinations, it is important to note that the limits given in Eq.(30) are completely independent of mixing angles. Therefore, if neutrinos are massive and mixed, their masses can be heavier than the limits given in Eq.(1). Of course, such heavy neutrinos, if stable, are not allowed by the well–known cosmological limits, $\sum_i m(\nu_i) \lesssim 20 \sim 30$ eV (with $h = (H_0 \text{ s Mpc})/100 \text{ Km} \simeq 0.5$, where H_0 is the Hubble constant). If unstable, the decay of neutrinos must be such that it should not disturb the standard nucleosynthesis scenario and should not violate the observed limits on cosmic electromagnetic wave backgrounds.

It is not possible to obtain absolute limits on the mixing angles, because they strongly depend on the input values of the masses. Examples of limits on the angles for definite values of the masses have been derived and reported in the previous Section.

To conclude, it is gratifying that the accuracy of the current data is already good enough to set limits on m_2 and m_3 which agree, in the order of magnitude, with the results from more involved kinematical determination, although the limits on the mixing angles are still rather poor. Further improvements of the data on τ, m_W and m_Z in the future may significantly improve the limits.

APPENDIX A: FORMULAS FOR PION AND LEPTON DECAYS

In this Appendix we present, for completeness, the expressions for the pion and lepton decays for the general case of three massive neutrinos with mixing. Radiative corrections to these processes are included with a brief comment.

1. Pion decay

The partial decay rate of the pion into two leptons is given by

$$\Gamma(\pi \longrightarrow l\bar{\nu}_l) = \frac{G^2 f_\pi^2 U_{ud}^2 m_\pi^3}{8\pi} \cdot \mathcal{R}_{\pi l} \cdot \sum_{i=1}^{3} |V_{li}|^2 P_i^{\pi l}, \tag{A1}$$

where the phase space–matrix element factor, $P_i^{\pi l}$, is given by

$$P_i^{\pi l} = \theta(m_\pi - m_l - m_i) \cdot [\delta_{l\pi}^2 + \delta_{i\pi}^2 - (\delta_{l\pi}^2 - \delta_{i\pi}^2)^2]\lambda^{1/2}(1, \delta_{l\pi}^2, \delta_{i\pi}^2). \tag{A2}$$

In the above, G is the Fermi constant, f_π is the pion decay constant, U is the mixing matrix in the quark sector,

$$\delta_{l\pi} = \frac{m_l}{m_\pi} \tag{A3}$$

$$\delta_{i\pi} = \frac{m_i}{m_\pi} \quad (i = 1, 2, 3) \tag{A4}$$

and λ is the standard kinematical function

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + xz). \tag{A5}$$

The quantity $\mathcal{R}_{\pi l}$ in Eq.(A1) describes the *leading* radiative corrections to the pion decay process [4,17] given by

$$\mathcal{R}_{\pi l} = \left[1 + \frac{2\alpha}{\pi} \ln \left(\frac{m_Z}{m_\rho} \right) \right] \left[1 + \frac{\alpha}{\pi} F(\delta_{l\pi}) \right]$$
$$\times \left\{ 1 - \frac{\alpha}{\pi} \left[\frac{3}{2} \ln \left(\frac{m_\rho}{m_\pi} \right) + C_1 + C_2 \frac{m_l^2}{m_\rho^2} \ln \left(\frac{m_\rho^2}{m_l^2} \right) + C_3 \frac{m_l^2}{m_\rho^2} + \dots \right] \right\} \tag{A6}$$

where

$$F(x) = 3 \ln x + \frac{13 - 19x^2}{8(1 - x^2)} - \frac{8 - 5x^2}{2(1 - x^2)^2} x^2 \ln x$$
$$- 2 \left(\frac{1 + x^2}{1 - x^2} \ln x + 1 \right) \ln(1 - x^2) + 2 \left(\frac{1 + x^2}{1 - x^2} \right) L(1 - x^2) \tag{A7}$$

Here, $m_\rho = 796$ MeV is the ρ meson mass, m_Z the Z boson mass and α is the fine structure constant. Also, in the above, $L(z)$ is defined by

$$L(z) = \int_0^z \frac{\ln(1 - t)}{t} dt. \tag{A8}$$

The first square bracket in Eq. (A6) represents the electromagnetic short–distance correction. Its value is slightly modified when higher–order effects and QCD corrections are taken into account, i.e. [17]

$$\left[1 + \frac{2\alpha}{\pi} \ln \left(\frac{m_Z}{m_\rho} \right) \right] \longrightarrow 1.0232. \tag{A9}$$

The second and the third brackets denote the QED corrections to the decay of a point-like pion [17,18]. Following a general practice, we neglect the terms with the C_i's, whose numerical values have large uncertainties [17]. Therefore, we use the following simplified expression

$$\mathcal{R}_{\pi l} = 1.0232 \cdot \left[1 + \frac{\alpha}{\pi} F(\delta_{l\pi}) \right] \cdot \left[1 - \frac{3\alpha}{2\pi} \ln \left(\frac{m_\rho}{m_\pi} \right) \right]. \tag{A10}$$

Note that $\mathcal{R}_{\pi l}$ depends both on the pion and the lepton mass.

2. Lepton decay

The partial decay rate of a lepton into three leptons is given by

$$\Gamma(l' \longrightarrow l\bar{\nu}_l\nu_{l'}) = \frac{G^2 m_{l'}^5}{192\pi^3} \cdot \mathcal{R}_{l'} \cdot \sum_{i,j=1}^{3} |V_{l'i}|^2 |V_{lj}|^2 P_{ij}^{l'l}, \tag{A11}$$

where

$$P_{ij}^{l'l} = \theta(m_{l'} - m_l - m_i - m_j) \cdot 2 \int_{x_{\min}}^{x_{\max}} (x^2 - 4\delta_{ll'}^2)^{1/2}\mathcal{M}dx \tag{A12}$$

and

$$\mathcal{M} = \mathcal{A}^{1/2}[x(1 + \delta_{ll'}^2 - x)\mathcal{A} + (2 - x)(x - 2\delta_{ll'}^2)\mathcal{B}], \tag{A13}$$

$$\mathcal{A} = \frac{1}{(1 + \delta_{ll'}^2 - x)^2}$$
$$\times[(1 + \delta_{ll'}^2 - x)^2 - 2(1 + \delta_{ll'}^2 - x)(\delta_{il'}^2 + \delta_{jl'}^2) + (\delta_{il'}^2 - \delta_{jl'}^2)^2], \tag{A14}$$

$$\mathcal{B} = \frac{1}{(1 + \delta_{ll'}^2 - x)^2}$$
$$\times[(1 + \delta_{ll'}^2 - x)^2 + (1 + \delta_{ll'}^2 - x)(\delta_{il'}^2 + \delta_{jl'}^2) - 2(\delta_{il'}^2 - \delta_{jl'}^2)^2]. \tag{A15}$$

In the above

$$\delta_{ll'} = \frac{m_l}{m_{l'}} \tag{A16}$$

$$\delta_{il'} = \frac{m_i}{m_{l'}} \quad (i = 1, 2, 3) \tag{A17}$$

$$x_{\min} = 2\delta_{ll'} \tag{A18}$$

$$x_{\max} = 1 + \delta_{ll'}^2 - (\delta_{il'} + \delta_{jl'})^2. \tag{A19}$$

The quantity $\mathcal{R}_{l'}$ describes the *leading* radiative corrections to the lepton decay process [19] which are given by

$$\mathcal{R}_{l'} = \left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]\left(1 + \frac{3}{5}\frac{m_{l'}^2}{m_W^2}\right), \tag{A20}$$

where m_W is the W boson mass. Note that $\mathcal{R}_{l'}$ depends only on the mass of the decaying lepton, but not on the mass of decay products.

REFERENCES

[1] R.E. Shrock, Phys. Lett. **96B**, 159 (1980); Phys. Rev. D **24**, 1232 (1981); Phys. Rev. D **24**, 1275 (1981).

[2] J. Bahcall and R.B. Curtis, Nuovo Cim. **21**, 422 (1961).

[3] T. Yamazaki *et al.*, in *Proceedings of the International Conference on High Energy Physics, Leipzig, 1984*, edited by A. Meyer and E. Wieckzorek (Desy–IFH, Zeuthen), p.262.

[4] Particle Data Group, Phys. Rev. D **50**, 1173 (1994).

[5] F. Boehm and P. Vogel, *Physics of massive neutrinos*, (Cambridge University Press, Cambridge, 1987); C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, (Contemporary Concepts in Physics – vol. 8, Harwood Academic Publisher, Chur, 1993).

[6] ALEPH Collaboration, D. Buskulic *et al.*, preprint CERN PPE/95–03.

[7] O.L.G. Peres, V. Pleitez and R. Zukanovich Funchal, Phys. Rev. D **50**, 513 (1994).

[8] Particle Data Group, Phys. Rev. D **45**, S1 (1992).

[9] BES Collaboration, J.Z. Bai *et al.*, Phys. Rev. Lett. **69**, 3021 (1992); ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. **202B**, 211 (1992).

[10] L. Maiani, Phys. Lett. **62B**, 183 (1976).

[11] A. Sirlin, Phys. Rev. D **22**, 971 (1980).

[12] The CDF Collaboration, Preprint FERMILAB–PUB–95/035–1995.

[13] *Reports of the working Group on precision calculations for the Z resonance*, ed. D. Bardeen, W. Hollik and G. Passarino, CERN Report CERN 95-03, March 1995.

[14] The CDF Collaboration, F. Abe et al., Phys. Rev. Lett. **74**, 2626 (1995).

[15] P.B. Renton, Proc. of the XVII International Symposium on Lepton–Photon Interactions, Beijing, P.R. China, 10-15 Aug 1995; K. Hagiwara, ibid.

[16] H. S. Chen, Proc. of the XVII International Symposium on Lepton–Photon Interactions, Beijing, P.R. China, 10-15 Aug 1995.

[17] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **71**, 3629 (1993).

[18] T. Kinoshita, Phys. Rev. Lett. **2**, 477 (1959).

[19] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **61**, 1815 (1988).

FIGURES

FIG.1 Allowed (thick solid lines) and forbidden (solid lines) intervals of m_3 by R's, $\overline{\Gamma}$'s and R's and $\overline{\Gamma}$'s combined, respectively, for 1992 and 1995 data sets ($m_1 \simeq m_2 \ll m_3$ is assumed). Mixing angles are varied over the interval $(0, \pi/2)$. Label 92 refers to the result of 1992 PDG data, without radiative corrections in the calculation; 92RC and 95RC denote the results of 1992 PDG and 1995 data sets with radiative corrections, respectively. The mass scale is in arbitrary units.

FIG.2 Allowed (dotted) regions in the β–γ plane for 1992 and 1995 data sets with $m_1 \simeq m_2 \simeq 0$ and $m_3 = 200$ MeV: (a) allowed region by R's alone (circles), $\overline{\Gamma}$'s alone (dots) and $R + \overline{\Gamma}$ combined (dark area) of 1992 PDG data, (b) allowed region by R's alone (circles) and by $\overline{\Gamma}$'s alone (dots) of 1992 PDG data with radiative corrections (the darker area is the region allowed both by the $\overline{\Gamma}$'s and the R's), (c) allowed region by R's alone (circles) of 1995 data with radiative corrections.

FIG.3 Allowed (dotted) region in the m_1–m_2 plane for 1995 data with radiative corrections. Both R's and $\overline{\Gamma}$'s constraints are imposed. Mixing angles are varied over the interval $(0, \pi/2)$.

FIG.4 Allowed (dotted) region in the m_2–m_3 plane for 1995 data with radiative corrections. Both R's and $\overline{\Gamma}$'s constraints are imposed. Mixing angles are varied over the interval $(0, \pi/2)$.

FIG.5 Allowed (dotted) region in the m_2–$\sin^2 \theta$ plane for 1995 data with radiative corrections and $m_1 = 5$ eV, $m_3 = 24$ MeV and $\beta = \gamma = 0$. The regions below the solid and the dashed lines are allowed by $\overline{\Gamma}$'s alone and R's alone, respectively.

FIG.6 Allowed (dotted) region in the m_3–$\sin^2 \gamma$ plane for 1995 data with radiative corrections and $m_1 = 5$ eV, $m_2 = 270$ KeV and $\theta = \beta = 0$. The regions below the solid and the dashed lines are allowed by $\overline{\Gamma}$'s alone and R's alone, respectively.

FIG.7 Allowed (dotted) region in the m_3–$\sin^2 \beta$ plane for 1995 data with radiative corrections, and $m_1 = 5$ eV, $m_2 = 270$ KeV and $\theta = \gamma = 0$. The regions below the solid and the dashed lines are allowed by $\overline{\Gamma}$'s alone and R's alone, respectively.

FIG.8 Allowed regions in the $\sin^2 \gamma$–$\sin^2 \beta$ plane for 1995 data with radiative corrections. The dotted area inside the solid line is for $m_1 \ll m_2 = 270$ KeV and $m_3 = 24$ MeV. The area on the left of the dashed line is for $m_1 \ll m_2 = 10$ KeV and $m_3 = 1$ MeV. Note the enlargement of the allowed region as masses decrease.

FIG.9 Allowed regions in the $\sin^2 \gamma$–$\sin^2 \theta$ plane for 1995 data with radiative corrections. Notation and interpretation are the same as in Fig.8.

FIG.10 Allowed regions in the $\sin^2 \beta$–$\sin^2 \theta$ plane for 1995 data with radiative corrections. Notation and interpretation are the same as in Fig.8.

TABLES

TABLE I. List of 1992 and 1995 data sets. The entries with asterisks denote data which have been improved by a significant amount.

		1992 data	1995 data
	m_e	$(0.51099906 \pm 0.00000015)\mathrm{MeV}$	same as 1992
	m_μ	$(105.658389 \pm 0.000034)\mathrm{MeV}$	same as 1992
*	m_τ	$(1784.1 \pm 3.6)\mathrm{MeV}$	$(1776.96 \pm 0.31)\mathrm{MeV}$
	m_π	$(139.5679 \pm 0.0007)\mathrm{MeV}$	$(139.56995 \pm 0.00035)\mathrm{MeV}$
	τ_μ	$(2.19703 \pm 0.00004) \cdot 10^{-6}$ s	same as 1992
*	τ_τ	$(305 \pm 6) \cdot 10^{-15}$ s	$(291.6 \pm 1.5) \cdot 10^{-15}$ s
*	m_Z	$(91.173 \pm 0.020) \cdot 10^3\mathrm{MeV}$	$(91.1884 \pm 0.0022) \cdot 10^3\mathrm{MeV}$
*	m_W	$(80.22 \pm 0.26) \cdot 10^3\mathrm{MeV}$	$(80.410 \pm 0.180) \cdot 10^3\mathrm{MeV}$
	G_μ	$(1.16639 \pm 0.00002) \cdot 10^{-11}\mathrm{MeV}^{-2}$	same as 1992
*	G	$(1.162 \pm 0.029) \cdot 10^{-11}\mathrm{MeV}^{-2}$	$(1.174 \pm 0.022) \cdot 10^{-11}\mathrm{MeV}^{-2}$
*	$\mathrm{BR}(\pi \longrightarrow e\bar{\nu}_e)$	$(1.218 \pm 0.014) \cdot 10^{-4}$	$(1.230 \pm 0.004) \cdot 10^{-4}$
	$\mathrm{BR}(\pi \longrightarrow \mu\bar{\nu}_\mu)$	0.9998782 ± 0.0000014	0.9998770 ± 0.0000004
	$\mathrm{BR}(\mu \longrightarrow e\bar{\nu}_e\nu_\mu)$	1	same as 1992
*	$\mathrm{BR}(\tau \longrightarrow e\bar{\nu}_e\nu_\tau)$	0.1793 ± 0.0026	0.1779 ± 0.0009
*	$\mathrm{BR}(\tau \longrightarrow \mu\bar{\nu}_\mu\nu_\tau)$	0.1758 ± 0.0027	0.1733 ± 0.0009
*	$\overline{\Gamma}^{\mu e}$	1.003 ± 0.050	0.983 ± 0.037
*	$\overline{\Gamma}^{\tau e}$	0.944 ± 0.053	0.979 ± 0.037
*	$\overline{\Gamma}^{\tau \mu}$	0.925 ± 0.052	0.954 ± 0.036
*	$R^{\pi e}_{\pi\mu}$	$(1.218 \pm 0.013) \cdot 10^{-4}$	$(1.230 \pm 0.004) \cdot 10^{-4}$
*	$R^{\mu e}_{\tau e}$	1.063 ± 0.028	1.0038 ± 0.0073
*	$R^{\tau e}_{\tau\mu}$	1.020 ± 0.022	1.0265 ± 0.0074